CALCULATION OF MAXIMUM TEMPERATURE DROPS AND THERMAL STRESSES IN HEATED CYLINDERS OF INFINITE LENGTH

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This paper presents analytical solutions for maximum temperature drops, and graphs and criterial relationships for the determination of maximum thermal stresses in heated cylinders with initial radial temperature variation.

Thermal stresses are one of the main limitations to the rate of heating of metals in furnaces. It has previously been shown [1], that when solids are heated some time elapses before the maximum temperature differences occur over section of the solid. According to majority of authors [2, 3] the maximum thermal stresses occur when the temperature differences are maximum. The volume of material with this maximum stress is then the parameter which limits its ratio of heating.

It is therefore of interest to determine the magnitude of the maximum temperature difference for various heating conditions and the time taken to establish this difference.

The present authors have investigated the heating of an infinite cylinder with a constant ambient temperature which case is frequently encountered in furnaces in steel mills.

In this case the temperature difference between the surface and axis of the cylinder of infinite length is given by [2]

$$\frac{\Delta t}{t_{\text{furn}}-t_0}=\sum_{n=1}^{\infty}\frac{2J_1(\mu_n)}{\mu_n\left[J_0^2(\mu_n)+J_1^2(\mu_n)\right]}\left[1-J_0(\mu_n)\right]\times\exp\left(-\mu_n^2\frac{a\tau}{R^2}\right). \tag{1}$$

The time at which the maximum temperature difference occurs is given by

$$\frac{\partial}{\partial\tau}\left(\frac{\Delta t}{t_{\text{furn}}-t_0}\right)=-\frac{a}{R^2}\sum_{n=1}^{\infty}B_n\mu_n^2\left[1-J_0'(\mu_n)\right]\exp\left(-\mu_n^2\frac{a\tau}{R^2}\right)=0, \tag{2}$$

where

$$B_n=\frac{2J_1(\mu_n)}{\mu_n\left[J_0^2(\mu_n)+J_1^2(\mu_n)\right]}.$$

If only the first two terms of equation are taken, a simpler equation is obtained

$$\mu_1^2B_1\left[1-J_0(\mu_1)\right]\exp\left(-\mu_1^2\frac{a\tau}{R^2}\right)+\mu_2^2B_2\left[1-J_0(\mu_2)\right]\times\exp\left(-\mu_2^2\frac{a\tau}{R^2}\right)=0,$$

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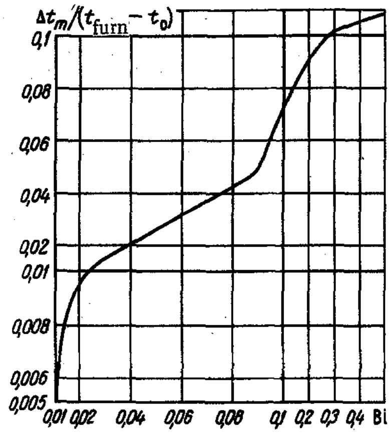

Fig. 1

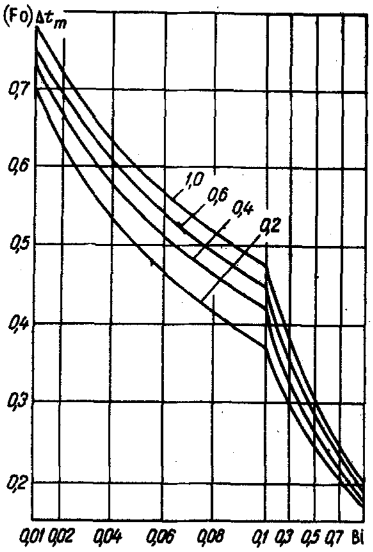

Fig. 2

Fig. 1. Relative maximum temperature difference $(\Delta t_m/(t_{furn} - t^0))$ versus Biot and Fourier numbers.

Fig. 2. Fourier number at maximum temperature drop over solid section $(Fo)_{\Delta t_m}$ versus Biot number (Bi) and relative initial temperature difference $(\Delta t_0/(t_{furn} - t_p^0))$. Numbers near curves are values of $\Delta t_0 /(t_{furn} - t_p^0)$.

when

$$\left(\frac{a\tau}{R^2}\right)_{\Delta t_m} = \frac{1}{\mu_1^2 - \mu_2^2} \ln \frac{\mu_1^2 B_1 \left[J_0(\mu_1) - 1\right]}{\mu_2^2 B_2 \left[1 - J_0(\mu_2)\right]} . \tag{3}$$

Equation (3) gives the value of τ at which the temperature difference Δt_m is a maximum for the given condition.

Substitution of the first value of $(a\tau/R^2)_{\Delta t_m}$ into equation (1) allows calculation of the maximum temperature differences occurring when the cylinder is heated

$$\Delta t_m \cong (t_{furn} - t_0) \sum_{n=1}^{n=2} B_n \left[1 - J_0(\mu_n)\right] \exp\left[-\mu_n^2 \left(\frac{a\tau}{R^2}\right)_{\Delta t_m}\right]. \tag{4}$$

The solution obtained is presented graphically (Fig. 1) in the form $(\Delta t_m/(t_{furn} - t_0)) = f\,[\mathrm{Bi}, (\mathrm{Fo})_{\Delta t_m}]$.

A similar relationship can be obtained when there is an initial temperature difference at the start of heating of the cylinder of infinite length.

In this case

$$t_s = t_{furn} + (t_p^0 - t_{furn}) \sum_{n=1}^{\infty} A_n J_0(\mu_n) \exp(-\mu_n^2 \mathrm{Fo}) - \Delta t_0 \sum_{n=1}^{\infty} B_n J_0(\mu_n) \exp(-\mu_n^2 \mathrm{Fo}), \tag{5}$$

$$t_{ax} = t_{furn} + (t_p^0 - t_{furn}) \sum_{n=1}^{\infty} A_n \exp(-\mu_n^2 \mathrm{Fo}) - \Delta t_0 \sum_{n=1}^{\infty} B_n \exp(-\mu_n^2 \mathrm{Fo}), \tag{6}$$

where

$$A_n = \frac{4 J_2(\mu_n)}{\mu_n^2 \left[J_0^2(\mu_n) + J_1^2(\mu_n)\right]},$$

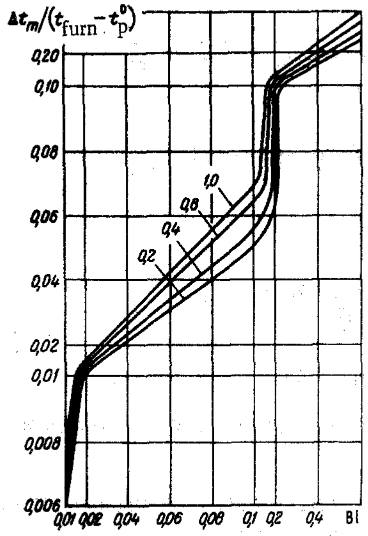

Fig. 3

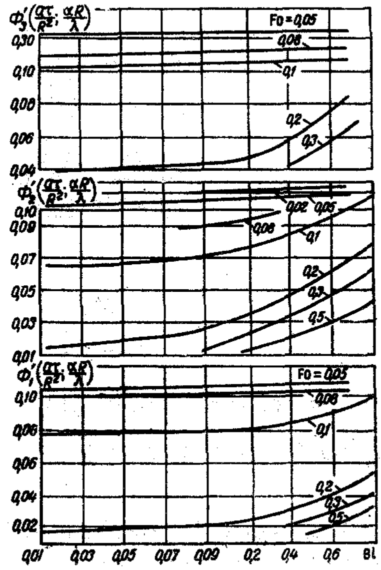

Fig. 4

Fig. 3. Relative maximum temperature drop in solid ($\Delta t_m/(t_{furn} - t_p^0)$) versus Biot number (Bi) and relative initial temperature difference ($\Delta t_0/(t_{furn} - t_p^0)$). Numbers near curves are values of $\Delta t_0/(t_{furn} - t_p^0)$.

Fig. 4. Functions Φ_1', Φ_2', Φ_3' versus Biot and Fourier numbers (Bi, Fo).

$$\frac{\Delta t}{t_{furn} - t_p^0} = \sum_{n=1}^{\infty} A_n \left[1 - J_0(\mu_n)\right] \exp(-\mu_n^2 \mathrm{Fo}) + \frac{\Delta t_0}{t_{furn} - t_p^0} \sum_{n=1}^{\infty} B_n \left[1 - J_0(\mu_n)\right] \exp(-\mu_n^2 \mathrm{Fo}). \quad (7)$$

To obtain $(\mathrm{Fo})_{\Delta t_m}$ the first value of equation (7) is found

$$\frac{\partial}{\partial \tau}(\Delta t) = -(t_p^0 - t_{furn}) \frac{a}{R^2} \sum_{n=1}^{\infty} \mu_n^2 A_n \left[1 - J_0(\mu_n)\right] \exp(-\mu_n^2 \mathrm{Fo}) - \Delta t_0 \sum_{n=1}^{\infty} \mu_n^2 B_n \left[1 - J_0(\mu_n)\right] \exp(-\mu_n^2 \mathrm{Fo}) = 0$$

or taking only the first two terms

$$\{(t_p^0 - t_{furn}) \mu_1^2 A_1 \left[1 - J_0(\mu_1)\right] \exp(-\mu_1^2 \mathrm{Fo}) + (t_p^0 - t_{furn}) \mu_2^2 A_2 \left[1 - J_0(\mu_2)\right] \exp(-\mu_2^2 \mathrm{Fo})\} = \{\Delta t_0 \mu_1^2 B_1 \left[1 - J_0(\mu_1)\right] \exp(-\mu_1^2 \mathrm{Fo}) + \Delta t_0 \mu_2^2 B_2 \left[1 - J_0(\mu_2)\right] \exp(-\mu_2^2 \mathrm{Fo})\}$$

when

$$(\mathrm{Fo})_{\Delta t_m} = \frac{1}{\mu_1^2 - \mu_2^2} \times \ln \frac{\mu_1^2 A_1 \left[1 - J_0(\mu_1)\right] + \frac{\Delta t_0}{t_{furn} - t_p^0} \mu_1^2 B_1 \left[1 - J_0(\mu_1)\right]}{\mu_2^2 A_2 \left[J_0(\mu_2) - 1\right] + \frac{\Delta t_0}{t_{furn} - t_p^0} \mu_2^2 B_2 \left[1 - J_0(\mu_2)\right]}. \quad (8)$$

Using (8) gives the maximum temperature difference from (7). Then

$$\frac{\Delta t_m}{t_{\text{furn}}-t_{\text{p}}^0} \simeq \sum_{n=1}^{n=2} A_n \left[1 - J_0(\mu_n)\right] \exp\left(-\mu_n^2 (\text{Fo})_{\Delta t_m}\right)$$

$$+ \frac{\Delta t_0}{t_{\text{furn}}-t_{\text{p}}^0} \sum_{n=1}^{n=2} B_n \left[1 - J_0(\mu_n)\right] \exp\left(-\mu_n^2 (\text{Fo})_{\Delta t_m}\right) \tag{9}$$

or

$$\frac{\Delta t_m}{t_{\text{furn}}-t_{\text{p}}^0} = f\left(\frac{\Delta t_0}{t_{\text{furn}}-t_{\text{p}}^0};\ \text{Bi};\ \text{Fo}_{\Delta t_m}\right).$$

Equations (8) and (9) are presented graphically (Figs. 2,3).

These equations can be used to calculate the maximum thermal stress occurring during heating of the cylinder.

To do so in practical cases of metal heating it is necessary only to evaluate stresses at the surface and on axis of the specimen since these are the maximum stresses [2].

If the initial temperature t_0 of the solid is uniform over its section, and the temperature of the ambient medium is t_{furn} = const for the whole of the heating process, then for the calculation of the radial σ_r, tangential σ_θ, and axial σ_z stresses the following slightly changed equations are used [2]

$$\sigma_r^0 = \sigma_\theta^0 = \frac{\beta E}{1-\nu} \cdot \frac{\Delta t}{\Phi_n - \Phi_c} \sum_{n=1}^{\infty} G_n \left[J_1(\mu_n) - \frac{\mu_n}{2} \right] \exp(-\mu_n^2 \text{Fo}), \tag{10}$$

$$\sigma_\theta^n = \sigma_z^n = \frac{\beta E}{1-\nu} \cdot \frac{\Delta t}{\Phi_n - \Phi_c} \sum_{n=1}^{\infty} G_n \left[2J_1(\mu_n) - \mu_n J_0(\mu_n)\right] \exp(-\mu_n^2 \text{Fo}), \tag{11}$$

$$\sigma_z^0 = \frac{\beta E}{1-\nu} \cdot \frac{\Delta t}{\Phi_n - \Phi_c} \sum_{n=1}^{\infty} G_n \left[2J_1(\mu_n) - \mu_n\right] \exp(-\mu_n^2 \text{Fo}), \tag{12}$$

where

$$G_n = \frac{2}{\mu_n^2} \cdot \frac{J_1(\mu_n)}{J_0^2(\mu_n) + J_1^2(\mu_n)}.$$

Calculations of the stresses have been made for cases in which the initial temperature differences along the infinite cylinder are known.

For such infinite cylinders the thermal stresses can generally be described [2] by the following equations

$$\sigma_r = \frac{\beta E}{1-\nu}\left(\frac{1}{2}\bar{t} - \frac{1}{2}\bar{t}_r\right), \tag{13}$$

$$\sigma_\theta = \frac{\beta E}{1-\nu}\left(\frac{1}{2}\bar{t} + \frac{1}{2}\bar{t}_r - t\right), \tag{14}$$

$$\sigma_z = \frac{\beta E}{1-\nu}(\bar{t} - t). \tag{15}$$

The values of $\bar{t}$ and $\bar{t}_r$ were obtained first for the cases being considered

$$\bar{t} = \frac{2}{R^2}\int_0^R t r dr = \frac{2}{R^2}\int_0^R \left[t_{\text{furn}} + (t_{\text{p}}^0 - t_{\text{furn}}) \sum_{n=1}^{\infty} A_n J_0\left(\mu_n \frac{r}{R}\right) \exp(-\mu_n^2 \text{Fo}) - \Delta t_0 \sum_{n=1}^{\infty} B_n J_0\left(\mu_n \frac{r}{R}\right) \exp(-\mu_n^2 \text{Fo})\right] r dr,$$

where the integral is in the form of $\int x J_0(ax)\,dx = (1/a) x J_1(ax) + \text{const}$ [4].

By iteration

$$\frac{\bar{t}-t_{\text{furn}}}{t_{\text{p}}^{0}-t_{\text{furn}}}=\sum_{n=1}^{\infty} A_n \frac{2J_1(\mu_n)}{\mu_n}\exp(-\mu_n^2\text{Fo})$$
$$-\frac{\Delta t_0}{t_{\text{p}}^{0}-t_{\text{furn}}}\sum_{n=1}^{\infty} B_n \frac{2J_1(\mu_n)}{\mu_n}\exp(-\mu_n^2\text{Fo}) \tag{16}$$

is obtained. Then by analogy

$$\bar{t}_r=\frac{2}{r^2}\int_0^r trdz=t_{\text{furn}}+(t_{\text{p}}^{0}-t_{\text{furn}})\sum_{n=1}^{\infty} 2A_n \frac{RJ_1\left(\mu_n\frac{r}{R}\right)}{r\mu_n}\exp(-\mu_n^2\text{Fo})$$
$$-\Delta t_0\sum_{n=1}^{\infty} 2B_n \frac{RJ_1\left(\mu_n\frac{r}{R}\right)}{r\mu_n}\exp(-\mu_n^2\text{Fo}). \tag{17}$$

Substituting the values of $\bar{t}$ and $\bar{t}_r$ into Equation (13) further iteration gives

$$\sigma_r=\frac{\beta E}{1-\nu}\left\{(t_{\text{p}}^{0}-t_{\text{furn}})\sum_{n=1}^{\infty} G_n\left[J_1(\mu_n)-\frac{R}{r}J_1\left(\mu_n\frac{r}{R}\right)\right]\exp(-\mu_n^2\text{Fo})\right.$$
$$\left.-\Delta t_0\sum_{n=1}^{\infty} M_n\left[J_1(\mu_n)-\frac{R}{r}J_1\left(\mu_n\frac{r}{R}\right)\right]\exp(-\mu_n^2\text{Fo})\right\}, \tag{18}$$

where

$$M_n=\frac{4}{\mu_n^3}\cdot\frac{J_2(\mu_n)}{J_0^2(\mu_n)+J_1^2(\mu_n)}.$$

The radial stress at the cylinder surface (r/R = 1)

$$\sigma_r^n=0.$$

The radial stress at the cylinder axis (r/R = 0)

$$\sigma_r^0=\frac{\beta E}{1-\nu}\left\{(t_{\text{p}}^{0}-t_{\text{furn}})\sum_{n=1}^{\infty} G_n\left[J_1(\mu_n)-\frac{\mu_n}{2}\right]\exp(-\mu_n^2\text{Fo})\right.$$
$$\left.-\Delta t_0\sum_{n=1}^{\infty} M_n\left[J_1(\mu_n)-\frac{\mu_2}{2}\right]\exp(-\mu_n^2\text{Fo})\right\}. \tag{19}$$

Substitution of (16) and (17) into Equation (14) and iteration gives

$$\sigma_\theta=\frac{\beta E}{1-\nu}\left\{(t_{\text{p}}^{0}-t_{\text{furn}})\sum_{n=1}^{\infty} G_n\left[J_1(\mu_n)+\frac{R}{r}J_1\left(\mu_n\frac{r}{R}\right)-\mu_nJ_0\right.\right.$$
$$\left.\times\left(\mu_n\frac{r}{R}\right)\right]\exp(-\mu_n^2\text{Fo})-\Delta t_0\sum_{n=1}^{\infty} M_n\left[J_1(\mu_n)+\frac{R}{r}J_1\left(\mu_n\frac{r}{R}\right)\right. \tag{20}$$
$$\left.\left.-\mu_nJ_0\left(\mu_n\frac{r}{R}\right)\right]\exp(-\mu_n^2\text{Fo})\right\}.$$

The tangential stress at the surface (r/R = 1) and on the axis (r/R = 0) of the cylinder are given by

$$\sigma_\theta^{\text{p}}=\frac{\beta E}{1-\nu}\left\{(t_{\text{p}}^{0}-t_{\text{furn}})\sum_{n=1}^{\infty} G_n\left[2J_1(\mu_n)-\mu_nJ_0(\mu_n)\right]\exp(-\mu_n^2\text{Fo})\right.$$

$$-\Delta t_0 \sum_{n=1}^{\infty} M_n \left[2J_1(\mu_n) - \mu_n J_0(\mu_n)\right] \exp(-\mu_n^2 \text{Fo}) \Big\}, \tag{21}$$

$$\sigma_\theta^0 = \frac{\beta E}{1-\nu} \Big\{ (t_p^0 - t_{\text{furn}}) \sum_{n=1}^{\infty} G_n \left[J_1(\mu_n) - \frac{\mu_n}{2}\right] \exp(-\mu_n^2 \text{Fo})$$

$$-\Delta t_0 \sum_{n=1}^{\infty} M_n \left[J_1(\mu_n) - \frac{\mu_n}{2}\right] \exp(-\mu_n^2 \text{Fo}) \Big\}. \tag{22}$$

Substitution of (16) and (17) into Equation (15) gives equation for the axial stress at the surface and on the axis of the cylinder

$$\sigma_z^n = \frac{\beta E}{1-\nu} \Big\{ (t_p^0 - t_{\text{furn}}) \sum_{n=1}^{\infty} G_n \left[2J_1(\mu_n) - \mu_n J_0(\mu_n)\right] \exp(-\mu_n^2 \text{Fo})$$

$$-\Delta t_0 \sum_{n=1}^{\infty} M_n \left[2J_1(\mu_n) - \mu_n J_0(\mu_n)\right] \exp(-\mu_n^2 \text{Fo}) \Big\}, \tag{23}$$

$$\sigma_z^0 = \frac{\beta E}{1-\nu} \Big\{ (t_p^0 - t_{\text{furn}}) \sum_{n=1}^{\infty} G_n \left[2J_1(\mu_n) - \mu_n\right] \exp(-\mu_n^2 \text{Fo})$$

$$-\Delta t_0 \sum_{n=1}^{\infty} M_n \left[2J_1(\mu_n) - \mu_n\right] \exp(-\mu_n^2 \text{Fo}) \Big\}. \tag{24}$$

The preceding equations can also be given in nondimensional form

$$\begin{gathered}
\frac{\sigma_r^0(1-\nu)}{\beta E(t_p^0 - t_{\text{furn}})} = \frac{\sigma_\theta^0(1-\nu)}{\beta E(t_p^0 - t_{\text{furn}})} = f_1\left(\frac{a\tau}{R^2};\ \frac{\alpha R}{\lambda}\right) - \frac{\Delta t_0}{t_p^0 - t_{\text{furn}}} \Phi_1'\left(\frac{a\tau}{R^2};\ \frac{\alpha R}{\lambda}\right), \\
\frac{\sigma_\theta^n(1-\nu)}{\beta E(t_p^0 - t_{\text{furn}})} = \frac{\sigma_z^n(1-\nu)}{\beta E(t_p^0 - t_{\text{furn}})} = f_2\left(\frac{a\tau}{R^2};\ \frac{\alpha R}{\lambda}\right) \\
- \frac{\Delta t_0}{t_p^0 - t_{\text{furn}}} \Phi_2'\left(\frac{a\tau}{R^2};\ \frac{\alpha R}{\lambda}\right), \\
\frac{\sigma_z^0(1-\nu)}{\beta E(t_p^0 - t_{\text{furn}})} = f_3\left(\frac{a\tau}{R^2};\ \frac{\alpha R}{\lambda}\right) - \frac{\Delta t_0}{t_p^0 - t_{\text{furn}}} \Phi_3'\left(\frac{a\tau}{R^2};\ \frac{\alpha R}{\lambda}\right).
\end{gathered} \tag{25}$$

If the first order solution of the above equation is presented graphically ([2]) then the second being of the form $\Phi_n(a\tau/R^2;\ \alpha R/\lambda;\ r/R)$ can also be represented graphically (Fig. 4) which is convenient for practical use.

The above equations for the stress in a cylinder can be written as functions of temperature difference. Equation (25) then becomes

$$\begin{gathered}
\frac{\sigma_r^0(1-\nu)}{\beta E\,\Delta t_0} = \frac{\sigma_\theta^0(1-\nu)}{\beta E\,\Delta t_0} = \frac{1}{\Phi_n^0 - \Phi_c^0} f_1\left(\frac{a\tau}{R^2};\ \frac{\alpha R}{\lambda}\right) - \Phi_1'\left(\frac{a\tau}{R^2};\ \frac{\alpha R}{\lambda}\right), \\
\frac{\sigma_\theta^n(1-\nu)}{\beta E\Delta t_0} = \frac{\sigma_z^n(1-\nu)}{\beta E\Delta t_0} = \frac{1}{\Phi_n^0 - \Phi_c^0} f_2\left(\frac{a\tau}{R^2};\ \frac{\alpha R}{\lambda}\right) - \Phi_2'\left(\frac{a\tau}{R^2};\ \frac{\alpha R}{\lambda}\right), \\
\frac{\sigma_z^0(1-\nu)}{\beta E\Delta t_0} = \frac{1}{\Phi_n^0 - \Phi_c^0} f_3\left(\frac{a\tau}{R^2};\ \frac{\alpha R}{\lambda}\right) - \Phi_3'\left(\frac{a\tau}{R^2};\ \frac{\alpha R}{\lambda}\right),
\end{gathered} \tag{26}$$

where

$$\Phi_n^0 = \frac{t_p^0 - t_{\text{furn}}}{t_0 - t_{\text{furn}}},\quad \Phi_c^0 = \frac{t_c^0 - t_{\text{furn}}}{t_0 - t_{\text{furn}}}.$$

Substitution of the maximum temperature difference Δt_m into (26) allows calculation of the maximum possible thermal stress for the particular heating condition, being considered.

NOTATION

$\Delta t = t_s - t_{ax}$	temperature difference between surface and cylinder axis, °C;
R	radius of cylinder, m;
t_0	initial temperature of solid, °C;
t_{furn}	temperature of furnace, °C;
a	thermal diffusivity, m^2/h;
τ	heating time, h;
J_1, J_2	Bessel functions of the first and zeroth orders;
μ_n	roots of characteristic equation $\mu J_1(\mu) = Bi\ J_0(\mu)$;
Bi	Biot number;
Fo	Fourier number;
$(Fo)_{\Delta t_m}$	Fourier number at maximum temperature difference;
t_p^0	temperature of cylinder surface at initial heating time-moment, °C;
Δt_0	initial temperature difference over solid section, °C;
$J_2(\mu_n)$	Bessel functions of the second order;
β	linear expansion coefficient;
ν	Poisson ratio;
$\bar{t}$	mean temperature over cylinder section, °C;
$\bar{t}_r$	mean temperature over a portion of cylinder section bounded with coordinate r, °C.

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